2101. Writing $m = 2k$, for $k \in \mathbb{N}$, we have a difference of two squares:

$$2^{2k} - 1 \equiv (2^k - 1)(2^k + 1).$$

But we are told that $2^{2k} - 1$ is prime. Hence, the lesser of these factors must be equal to 1. So $k = 1$, which gives $m = 2$. □

2102. The intersections are at $\sqrt{x} = \sqrt[3]{x}$. Dividing by $\sqrt[3]{x}$, while noting the root $x = 0$, we get $x^{\frac{1}{6}} = 1$. Hence, $x = 0, 1$. Over the domain $[0, 1]$, the cube root graph is above the square root graph, so the area is

$$\begin{aligned} A &= \int_0^1 x^{\frac{1}{3}} - x^{\frac{1}{2}}\,dx \\ &= \left[\tfrac{3}{4}x^{\frac{4}{3}} - \tfrac{2}{3}x^{\frac{3}{2}}\right]_0^1 \\ &= \left(\tfrac{3}{4} - \tfrac{2}{3}\right) - (0) \\ &= \tfrac{1}{12}. \end{aligned}$$

2103. (a) Differentiating twice,

$$\mathbf{a} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \text{ ms}^{-2}.$$

The magnitude of the acceleration is

$$\begin{aligned} a &= \sqrt{0^2 + (-2)^2 + 4^2} \\ &= \sqrt{20} \text{ ms}^{-2} \end{aligned}$$

(b) Constant acceleration gives a parabolic path, irrespective of the dimensionality. This can be seen explicitly by rewriting the position as

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} t^2.$$

The initial position doesn't affect the shape of the path. The second and third vectors in the expression above, which are initial velocity and acceleration, then define a 2D plane, in which the path is parabolic.

2104. Set the side length to 1. Now, the interior angle of a pentagon is given by $180 - \frac{360}{5} = 108°$. So, if we construct a triangle of three adjacent vertices, the square of its longer side is given by the cosine rule to be

$$b = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cos 108°.$$

Hence, $a : b$ is $1 : 2(1 - \cos 108°)$ as required.

2105. Solving the inequalities, $z < \ln 1000 = 4.605...$ and $z > \log_2 7 = 2.807....$ We are looking for the set of integers which simultaneously satisfy both. This is $\{3, 4\}$.

2106. (a) i. The probability that a bag of fertiliser is underweight is $\mathbb{P}(M < 1) = 0.309$ (3sf).

ii. Using the probability from (a),

$$\begin{aligned} &\mathbb{P}(\text{at least one bag is underweight}) \\ &\equiv 1 - \mathbb{P}(\text{neither bag is underweight}) \\ &= 1 - 0.691^2 \\ &= 0.522 \text{ (3sf)}. \end{aligned}$$

(b) In ii., we are assuming that the masses of the two bags are independent of one another.

2107. Completing the square, the parabola is

$$y = -\left(x - \tfrac{1}{2}\right)^2 + \tfrac{29}{4}.$$

The new parabola is positive and has the same vertex, so it must have equation

$$y = \left(x - \tfrac{1}{2}\right)^2 + \tfrac{29}{4}.$$

2108. A contact force can be considered as the vector sum of a reaction force perpendicular to the surface and a frictional force parallel to it. If the cylinder is smooth, there can be no frictional force, hence the contact force must be all reaction. It must therefore act along a radius of the cylinder, which must pass through the axis of symmetry.

2109. The number of arrangements of $\text{A}_1\text{R}_1\text{R}_2\text{A}_2\text{NGED}$ is $8!$. Of these, there are $2! \times 2! = 4$ which spell the word ARRANGED, since we can place the As either way round, and the Rs either way around. Hence,

$$p = \frac{2! \times 2!}{8!}, \text{ as required.}$$

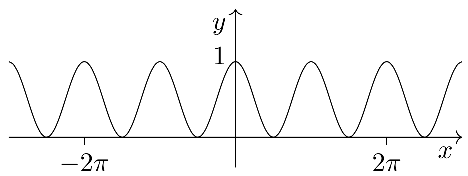
2110. The relevant double-angle formula is

$$\cos 2x \equiv 1 - 2\sin^2 x.$$

Rearranging to make $\sin^2 x$ the subject, the graph in question is

$$y = \tfrac{1}{2}(\cos 2x + 1).$$

This can be considered as a transformation of $y = \cos x$: stretch factor $\frac{1}{2}$ in the x direction, and translation by vector $\mathbf{j}$ followed by stretch factor $\frac{1}{2}$ in the y direction. This gives



2111. We rearrange and square to give

$$\sec^2 t = \left(\frac{p-a}{b}\right)^2,$$
$$\tan^2 t = \left(\frac{q-c}{d}\right)^2.$$

Substituting these expressions into the second Pythagorean trig identity $1+\tan^2 x \equiv \sec^2 x$ gives

$$1 + \left(\frac{q-c}{d}\right)^2 = \left(\frac{p-a}{b}\right)^2.$$

2112. Equating the ratios of successive terms,

$$\begin{aligned} \frac{b}{b-4} &= \frac{2b+6}{b} \\ \Longrightarrow b^2 &= (2b+6)(b-4) \\ \Longrightarrow b^2 - 2b - 24 &= 0 \\ \Longrightarrow b &= -4, 6. \end{aligned}$$

So, the common ratio is 3 or $\frac{1}{2}$.

2113. (a) The product and chain rules give

$$\begin{aligned} \frac{dy}{dx} &= 2x(3x-2)^3 + 9(x^2+1)(3x-2)^2 \\ &\equiv (15x^2 - 4x + 9)(3x-2)^2. \end{aligned}$$

The second derivative is

$$\frac{d^2y}{dx^2} = 2(90x^2 - 48x + 31)(3x-2).$$

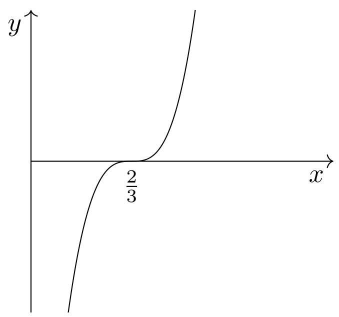
The first and second derivatives are zero at the x intercept $x = 2/3$. And, since $(3x-2)$ is a single factor in the second derivative, we also know that the second derivative changes sign at $x = 2/3$. So, the x intercept is a stationary point of inflection.

— Alternative Method —

Since the factor $(3x-2)$ is cubed, there is a triple root at $x = 2/3$. At a triple root, the graph is necessarily both stationary (repeated root) and inflected (root of odd multiplicity greater than 1).

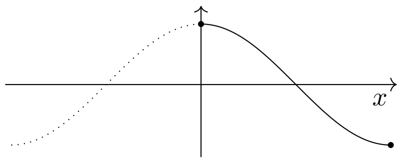
(b) In the first derivative, the quadratic factor $15x^2 - 4x + 9$ has $\Delta = -524 < 0$. So, the first derivative is only zero at $x = 2/3$.

(c) Since (x^2+1) has no real roots, there is one x intercept, the stationary point of inflection at $x = 2/3$. The equation is a positive quintic, so the curve is

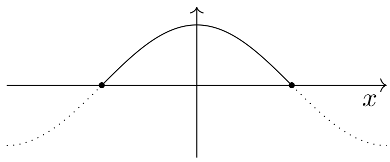


2114. T will be at its maximum if the other three forces are parallel, requiring $T = 18$. On the other hand, T will be at its minimum if the 3 and 5 N forces act to oppose the 10 N force, requiring $T = 2$. So, the set of possible values is $T \in [2, 18]$.

2115. The cosine function has period 2π. So, an interval of the form $[a, a+\pi]$ contains half a period. This can produce the full range $y \in [-1, 1]$ of the cosine function, at e.g. $x \in [0, \pi]$:

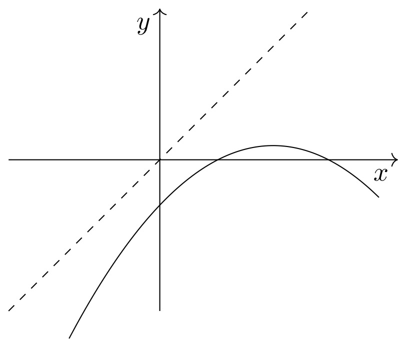


The narrowest range it can produce is at e.g. $x \in [-\pi/2, \pi/2]$, giving $[0, 1]$.



Hence, the greatest and least values of $q - p$ are 2 and 1.

2116. A root where $f(x) = 0$ is an intersection of $y = f(x)$ with the x axis; a fixed point where $f(x) = x$ is an intersection of $y = f(x)$ with the line $y = x$. So, any curve which crosses the x axis but not $y = x$ will do, such as



2117. (a) The two points have coordinates $(0, -1)$ and $(2, 1)$. Hence, the gradient of the chord is $m = 1$. Using $y - y_1 = m(x - x_1)$, its equation, therefore, is $y + 1 = x$ or $y = x - 1$.

(b) The first derivative is $\frac{dy}{dx} = 4x^3 - 12x^2 + 10x - 1$. Equating this to the gradient of the chord, we have $4x^3 - 12x^2 + 10x - 1 = 1$. A cubic solver gives

$$x = 1, 1 \pm \tfrac{\sqrt{2}}{2}.$$

Checking the y coordinates of these points, we see that $(1, 0)$ lies on the chord $y = x - 1$. Since it has gradient 1, the chord must be tangent to the curve at this point.

2118. Using $\log_a^2 b^2 = \log_a b$, we can rewrite everything in terms of base 4, and then simplify:

$$\begin{aligned} &\log_2 y - \log_4 y = \log_2 x + \log_4 x \\ \Longrightarrow\ &\log_4 y^2 - \log_4 y = \log_4 x^2 + \log_4 x \\ \Longrightarrow\ &\log_4 \tfrac{y^2}{y} = \log_4(x^2 \cdot x) \\ \Longrightarrow\ &y = x^3. \end{aligned}$$

2119. Since height is assumed to be continuous, there are exactly fifty people above the population median height. The probability changes as individuals are selected:

$$\begin{aligned} p &= \tfrac{50}{100} \times \tfrac{49}{99} \times \tfrac{48}{98} \times \tfrac{47}{97} \times \tfrac{46}{96} \\ &= 0.0281 \text{ (3sf)}. \end{aligned}$$

2120. The binomial expansion gives

$$(\sqrt{x} \pm x)^3 \equiv x^{\frac{3}{2}} \pm 3x^2 + 3x^{\frac{5}{2}} \pm x^3.$$

So, the equation is

$$\begin{aligned} &2x^{\frac{3}{2}} + 6x^{\frac{5}{2}} = 4x^{\frac{1}{2}} \\ \Longrightarrow\ &x^{\frac{1}{2}}(3x^2 + x - 2) = 0 \\ \Longrightarrow\ &x^{\frac{1}{2}}(3x - 2)(x + 1) = 0 \\ \Longrightarrow\ &x = 0, \tfrac{2}{3}, -1. \end{aligned}$$

However, the square root function takes positive reals, so the solution set is $\{0, 2/3\}$.

2121. This is correct. When you press the accelerator of a resting car, the wheels rotate, or try to. The driving force so produced is a frictional interaction between the wheels and the ground. The ground pushes the wheels forwards; this is usually called the *driving force*. By NIII, this must produce an equal and opposite force acting backwards on the ground. It is this force which kicks up dust and grit in wheel-spin.

2122. Assume, for a contradiction, that $f'(a_1) = g'(a_1)$. This means that the parabolae $y = f(x)$ and $y = g(x)$ are tangent to each other at $x = a_1$. Hence, the equation $f(x) = g(x)$ has a double root at $x = a_1$. So, since it is a quadratic equation, it cannot have any other roots. But we are told that $a_2 \neq a_1$ is also a root. This is a contradiction. Hence, $f'(a_1) \neq g'(a_1)$. The same argument holds for a_2. □

2123. (a) With μ as the population mean growth rate, the hypotheses are

$$\begin{aligned} \text{Null} \quad &H_0 : \mu = 1.1802 \\ \text{Alternative} \quad &H_1 : \mu \neq 1.1802 \end{aligned}$$

(b) This is a two-tail test at the 1% level. So, we need a 0.5% critical region at each tail. This is a critical value of 2.5758 standard deviations from the mean. For a sample of 24, the critical $\bar{R}$ values are

$$1.1802 \pm 2.5758 \times \sqrt{\frac{0.0025}{24}}.$$

So, the c.r. is $(-\infty, 1.1539) \cup (1.2065, \infty)$.

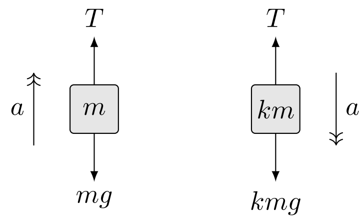
(c) The test statistic 1.1624 does not lie within the critical region. Hence, there is insufficient evidence, at the 1% level, to reject the null hypothesis. Substance P216 seems to have no noticeable effect on mould growth.

2124. The statement is true. If α is a fixed point of $g(x)$, then $g(\alpha) = \alpha$. Applying g^{-1} to both sides gives $\alpha = g^{-1}(\alpha)$, which tells us that α is a fixed point of g^{-1} as well. And the same argument works in reverse. So, a function and its inverse have the same set of fixed points. QED.

— Alternative Method —

Graphically, fixed points of f are intersections of the graph $y = f(x)$ and the line $y = x$. And the graph $y = f^{-1}(x)$ is a reflection of $y = f(x)$ in the line $y = x$. The graphs $y = f(x)$ and $y = f^{-1}(x)$ must therefore have the same set of intersections with $y = x$, so f and f^{-1} must have the same set of fixed points. QED.

2125. (a) Force diagrams:



The equations of motion are $T - mg = ma$ and $kmg - T = kma$. Adding these, we get $(k-1)mg = (k+1)ma$, so

$$a = \frac{k-1}{k+1}g \equiv \frac{k+1-2}{k+1}g \equiv \left(1 - \frac{2}{k+1}\right)g.$$

(b) To model the accelerations of the masses as equal (without doing which we cannot solve the problem), we must assume that the string is inextensible.

(c) The limit for large values of k is

$$\lim_{k\to\infty} \left(1 - \frac{2}{k+1}\right) g = g.$$

So, as the ratio between the masses gets large, the system approaches freefall acceleration.

2126. Since the curve is a quadratic, a stationary point on the x axis implies a double root there. Hence, the factors $(ax + b)$ and $(cx + d)$ must be scalar multiples of each other. In algebra, this is $\frac{c}{a} = \frac{d}{b}$, which we can rearrange to $ad - bc = 0$.

2127. Since A and B are independent, we know that $\mathbb{P}(A \cap B) = \mathbb{P}(A) \times \mathbb{P}(B)$. If they are also mutually exclusive, then then probability of both occurring is zero, hence $\mathbb{P}(A) \times \mathbb{P}(B) = 0$. This requires either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

2128. (a) $\displaystyle\lim_{x\to 0} \frac{2^x}{2^x + 1} = \frac{1}{2}$,

(b) $\displaystyle\lim_{x\to 1} \frac{2^x}{2^x + 1} = \frac{2}{3}$,

(c) $\displaystyle\lim_{x\to \infty} \frac{2^x}{2^x + 1} = \lim_{x\to \infty} \frac{1}{1 + 2^{-x}} = 1$.

2129. This isn't true: we use the term "increasing" to mean *locally* increasing, as defined by $\frac{dy}{dx} < 0$, not *globally* increasing, as suggested in the statement given. Consider, as a counterexample,

$$\text{f}(x) = -\frac{1}{x},$$

defined over the domain $\mathbb{R} \setminus \{0\}$, with $a = -2$ and $b = 2$. The first derivative is $\text{f}'(x) = x^{-2}$, which is positive at every point in the domain $\mathbb{R} \setminus \{0\}$. So, the function is *locally* increasing. However, it isn't *globally* increasing as suggested: $\text{f}(-2) = 1/2$ is greater than $\text{f}(2) = -1/2$.

2130. (a) The definitions give us

$$\begin{aligned} &a^2 \oplus b^2 - 2(a \otimes b) \\ = \ &a^4 + b^4 - 2a^2b^2 \\ \equiv \ &\left(a^2 - b^2\right)^2 \\ \equiv \ &(a + b)^2(a - b)^2. \end{aligned}$$

(b) The LHS is $\left(a^2+b^2\right)^2 c^2$. The RHS is $a^4c^4+b^4c^4$. Multiplying out the LHS gives

$$a^4c^4 + b^4c^4 + 2a^2b^2c^2.$$

This differs from the RHS by $2a^2b^2c^2$. For non-zero a, b, c this is non-zero, so the two sides are not identical.

— Alternative Method —

As a counterexample, consider $a = b = c = 1$. The LHS is

$$(1 \oplus 1) \otimes 1 = 2 \otimes 1 = 4.$$

The RHS, however, is

$$(1 \otimes 1) \oplus (1 \otimes 1) = 1 \oplus 1 = 2.$$

So, the sides are not identical.

2131. (a) A "census" is a sample which is, or attempts to be, the entire population. So, we can use ρ (*rho*, the Greek letter r), which represents the population, as opposed to sample, correlation coefficient.

(b) There is a clear positive correlation between the variables w and h. It is medium strength, with a correlation coefficient ρ in the broad vicinity of $+0.5$.

(c) The pupil goes against the pattern of positive correlation, being very tall but also very light. So, reinstating them will reduce the correlation coefficient ρ slightly.

2132. Integrating, $\text{F}(x) = \frac{1}{2}ax^2 + bx + c$. This gives the LHS as $(8a + 4b) - (0) \equiv 8a + 4b$. The RHS is

$$\begin{aligned} &2\left(\tfrac{9}{2}a + 3b\right) - 2\left(\tfrac{1}{2}a + b\right) \\ \equiv \ &8a + 4b. \end{aligned}$$

The two sides are equivalent, as required.

2133. The possibility space contains $6^3 = 216$ equally likely outcomes. Of these, $(6, 6, 6)$ and the three orders of $(5, 6, 6)$ sum to at least 17. Hence, the probability is $\frac{4}{216} = \frac{1}{54}$.

2134. This is only true is some circumstances, depending on the entity to which NII is being applied.

- If we are applying NII to the system of both objects together, which we may do since they are accelerating together, then the statement does hold: the internal forces form NIII pairs, which cancel out.
- However, if we are applying NII to one of the objects on its own, then only one of the NIII pair appears, so it cannot be ignored. In this case the statement doesn't hold.

2135. Starting with the LHS, we have

$$\begin{aligned} T_{a+b} &= \tfrac{1}{2}(a + b)(a + b + 1) \\ &= \tfrac{1}{2}(a^2 + b^2 + 2ab + a + b + 1) \\ &= \tfrac{1}{2}(a^2 + a) + \tfrac{1}{2}b^2 + b) + ab \\ &= \tfrac{1}{2}a(a + 1) + \tfrac{1}{2}b(b + 1) + ab \\ &= T_a + T_b + ab, \quad \text{as required.} \end{aligned}$$

2136. Differentiating twice,

$$\begin{aligned} &\text{f}(x) = a(x - b)^3 \\ \Longrightarrow \ &\text{f}'(x) = 3a(x - b)^2 \\ \Longrightarrow \ &\text{f}''(x) = 6a(x - b). \end{aligned}$$

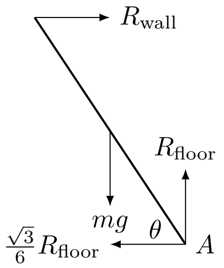
Consider now the respective signs of $\text{f}(x)$ and $\text{f}''(x)$. The scale factor between the two is $6(x-b)^2$, which is necessarily positive. Hence, the function and its second derivative must have the same sign. This is the required result.

2137. The statement is not true. There are two types of counterexample, as follows:

- If the equations are scalar multiples of each other, such as $x + y = 1$ and $2x + 2y = 2$, then the solution set contains infinitely many points.
- Alternatively, if the equations describe a pair of parallel lines, such as $x + y = 1$ and $x + y = 2$, then there are no intersections and the solution set is empty.

2138. (a) The ladder is in equilibrium, with the base as far away from the wall as possible, so friction must be maximal: $F_{\text{max}} = \mu R$.

(b) Using $F_{\text{max}} = \frac{\sqrt{3}}{6} R_{\text{floor}}$, the force diagram is



Taking the ladder to have length 2 m, our three equations are

$$\begin{aligned}
\updownarrow &: R_{\text{floor}} - mg = 0 \\
\leftrightarrow &: R_{\text{wall}} - \tfrac{\sqrt{3}}{6} R_{\text{floor}} = 0 \\
\overset{\frown}{A} &: R_{\text{wall}} \cdot 2 \sin\theta - mg \cdot 1 \cos\theta = 0.
\end{aligned}$$

(c) The first two equations give $R_{\text{floor}} = mg$ and $R_{\text{wall}} = \frac{\sqrt{3}}{6} mg$. Substituting into the moments equation,

$$\begin{aligned}
&\tfrac{\sqrt{3}}{6} mg \cdot 2 \sin\theta - mg \cdot 1 \cos\theta = 0 \\
\Longrightarrow\ &\tfrac{2\sqrt{3}}{6} \sin\theta = \cos\theta \\
\Longrightarrow\ &\tan\theta = \tfrac{6}{2\sqrt{3}}.
\end{aligned}$$

This gives $\theta = 60°$.

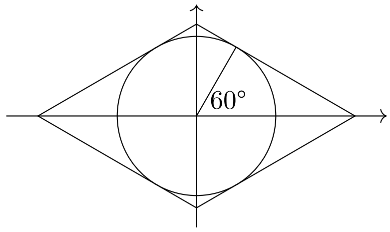
2139. Even through the two integrals look identical, they retain some flexibility. Each is an anti-derivative, so generates a $+c$. However, there is no reason why these $+c$'s should be the same. So, in fact, if we take $\text{F}'(x) = \text{f}(x)$, then the equation reads

$$\text{F}(x) + c_1 = 1 + \text{F}(x) + c_2.$$

In other words, $c_1 = 1 + c_2$, which is possible.

2140. This is not true. By the chain rule, the derivative of $\text{f}(1 - x)$ is $-\text{f}'(1 - x)$, which is the negative of the suggested result. That factor of -1 is the derivative of the inside function $(1 - x)$.

2141. Adding in lines of symmetry and a radius to a point of tangency, the scenario is



Call the radius of the circle 1; its area is $A_{\text{circ}} = \pi$. Rhombus side length is $\tan 60° + \tan 30° = \frac{4\sqrt{3}}{3}$. The area of the rhombus is that of two equilateral triangles of this side length, which is

$$A_{\text{rhom}} = 2 \cdot \tfrac{1}{2} \cdot \tfrac{4\sqrt{3}}{3} \cdot \tfrac{4\sqrt{3}}{3} \cdot \sin 60° = \tfrac{8\sqrt{3}}{3}.$$

Multiplying up by 3, the ratio $A_{\text{rhom}} : A_{\text{circ}}$ is then $8\sqrt{3} : 3\pi$, as required.

2142. Multiplying by the bottom of the LHS, which is the square of a difference of two squares,

$$\begin{aligned}
&\frac{1}{(x^2 - 1)^2} \equiv \frac{P}{x^2 - 1} + \frac{Q}{x^2 + 1} \\
\Longrightarrow\ &1 \equiv P(x^2 + 1) + Q(x^2 - 1).
\end{aligned}$$

We can now equate coefficients:

$$\begin{aligned}
x^0 &: 1 = P - Q, \\
x^2 &: 0 = P + Q.
\end{aligned}$$

This gives $P = \frac{1}{2}$, $Q = -\frac{1}{2}$.

2143. The largest angle θ is opposite $n + 2$. Cosine rule:

$$\begin{aligned}
\cos\theta &= \frac{n^2 + (n + 1)^2 - (n + 2)^2}{2n(n + 1)} \\
&\equiv \frac{n^2 - 2n - 3}{2n(n + 1)} \\
&\equiv \frac{(n - 3)(n + 1)}{2n(n + 1)} \\
&\equiv \frac{n - 3}{2n}.
\end{aligned}$$

Since $n > 3$, this fraction is positive. It is also less than 1, since $2n > n - 3$. Hence, θ is acute. QED.

2144. (a) $y = \text{f}(x)$ has a stationary point of inflection at $(0, 4)$. Hence, the curve $y = \text{f}(x) - 4$, which is also a cubic, must have a stationary point of inflection at the origin. This means it must have a triple root at zero, i.e. a triple factor of x. Hence, $\text{f}(x) - 4 = ax^3$. Rearranging gives $\text{f}(x) = ax^3 + 4$.

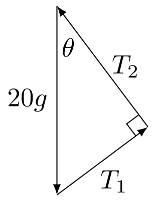
(b) Substituting $(2, 28)$ gives $28 = 8a + 4$, so $a = 3$. The equation of the graph is $y = 3x^3 + 4$.

2145. This is a quadratic in $\sin x$. We can factorise it to $(\sin x + 4)(\sin x - 3) = 0$. But, since the range of the sine function is $[-1, 1]$, neither factor can be made zero. Hence, the equation has no roots.

2146. In each case, the outputs are scaled down to ensure that the magnitude of the output is less than 1. The odd power leaves the range negative/positive, while the even power makes it solely positive. The ranges are

(a) $(-1, 1)$,
(b) $[0, 1)$.

2147. Since $(1.5, 2, 2.5)$ is similar to a $(3, 4, 5)$ triangle, the angle between the two ropes is 90°. Hence, the triangle of forces is as follows, where $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.



Trigonometry gives $T_1 = 12g$ and $T_2 = 16g$. The tensions are 117.6 N and 156.8 N.

2148. This is not a well defined function. The codomain is $\mathbb{Q}^+$, which is the set of positive rationals. But, for example, $\text{h}(2) = \sqrt{2}$ is irrational.

2149. The factor of $\frac{1}{n^2}$ is constant as far as the index i is concerned. Hence, we can take it out of the sum, although not out of the limit:

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} i.$$

We can then use the standard result for the sum of the first n integers, which brings

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n^2}\cdot \tfrac{1}{2}n(n+1)\\ =\ &\tfrac{1}{2}\lim_{n\to\infty}\frac{n}{n+1}.\end{aligned}$$

We can now take the limit, which is 1, yielding the required result.

— Nota Bene —

The rule determining whether a multiplier can be taken out is exactly the same for a sigma sum, an integral, a limit, or indeed for a set of terms. The question is whether the multiplier is constant (a common factor). The common factor k *can* be taken out in these:

$$\sum_{i=0}^{n} ki, \qquad \int_1^2 kx\,dx, \qquad \lim_{h\to 0}\frac{k(1+h)}{2}.$$

But k *cannot* be taken out in these:

$$\sum_{k=0}^{n} ki, \qquad \int_1^2 kx\,dk, \qquad \lim_{k\to 0}\frac{k(1+h)}{2}.$$

2150. We know, since $\mathbb{P}(A \mid B) = \mathbb{P}(A' \mid B)$, that A and B are independent. So, B gives no information about the probability of A. This means that

$$\begin{aligned}\mathbb{P}(A \mid B) &= \mathbb{P}(A),\\ \mathbb{P}(A' \mid B) &= \mathbb{P}(A').\end{aligned}$$

So, $\mathbb{P}(A) = \mathbb{P}(A')$, from which we get $\mathbb{P}(A) = 1/2$. Also, $\mathbb{P}(B) = 1/2$. This gives the probability of both as $\mathbb{P}(A \cap B) = 1/4$.

2151. The standard formula gives

$$\frac{d}{dx}\left(\frac{1}{x}\right) \equiv \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}.$$

Multiplying top and bottom by $x(x+h)$, we can then simplify as follows:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{x}\right) &\equiv \lim_{h\to 0}\frac{x-(x+h)}{hx(x+h)}\\ &\equiv \lim_{h\to 0}\frac{-h}{hx(x+h)}\\ &\equiv \lim_{h\to 0}\frac{-1}{x(x+h)}\\ &\equiv \frac{-1}{x^2}, \text{ as required.}\end{aligned}$$

2152. The second derivative is

$$\frac{d^2y}{dx^2} = 12x^2 - 6x + 2.$$

This is a positive quadratic with discriminant

$$\Delta = 36 - 4\cdot 12\cdot 2 = -60 < 0.$$

Hence, the second derivative is positive for all x, i.e. the curve is convex everywhere.

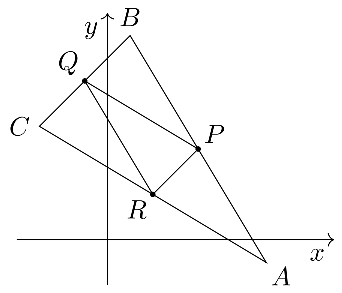
2153. Using a standard integration result and the reverse chain rule,

$$\int \frac{1}{\cos^2 5x}\,dx = \int \sec^2 5x\,dx = \tfrac{1}{5}\tan 5x + c.$$

2154. The greatest vertical displacement occurs when the vertical velocity is zero. Using $v^2 = u^2 + 2as$,

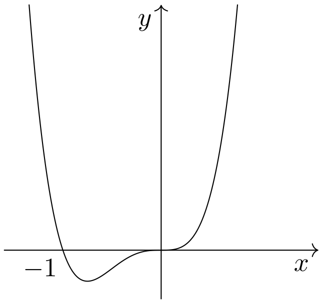
$$\begin{aligned}&0 = (u\sin\theta)^2 + 2\cdot -g\cdot h\\ \Longrightarrow\ &h = \frac{u^2\sin^2\theta}{2g}, \text{ as required.}\end{aligned}$$

2155. Labelling the midpoints as P, Q, R respectively, the situation is as follows:



Since P, Q, R are the midpoints of the relevant sides, $\triangle APR$ is similar to $\triangle ABC$, scale factor 2. Hence, A can be found by translating P by vector $\overrightarrow{QR}$. This gives $A : (7, -1)$. The same method yields $B : (1, 9)$ and $C : (-3, 5)$.

2156. This is a positive quartic. Factorising it, we have $y = x^3(x+1)$, which shows that it has a triple root at $x = 0$ and a single root at $x = -1$. Hence, the curve crosses the x axis at $x = -1$, while it has a stationary point of inflection at the origin:



2157. (a) The factor $(x-3)$ could be positive or negative, depending on the value of x. And multiplying by a negative number would switch the sign of the inequality.

(b) The boundary equation is $\frac{x-2}{x-3} = 1$. The top and bottom of the fraction cannot be equal, so this has no roots.

(c) We can rewrite as

$$\frac{x-2}{x-3} \equiv \frac{x-3+1}{x-3} \equiv 1 + \frac{1}{x-3}.$$

Subtracting 1, the inequality is now

$$\frac{1}{x-3} \leq 0.$$

For the fraction to be negative, we require $x < 3$. So, the solution set is $(-\infty, 3)$.

2158. (a) The situation is symmetrical in $y = x$. So, it is easiest to solve $x^2 - x = x$, which gives intersections at the origin and $(2, 2)$.

(b) Again using symmetry, we can find the area between $y = x^2 - x$ and $y = x$ by calculating the definite integral of the vertical distance:

$$\begin{aligned} A &= \int_0^2 x - (x^2 - x)\,dx \\ &= \left[x^2 - \tfrac{1}{3}x^3\right]_0^2 \\ &= \left(2^2 - \tfrac{1}{3}2^3\right) - (0) \\ &= \tfrac{4}{3}. \end{aligned}$$

Hence, the area enclosed is $2 \times \frac{4}{3} = \frac{8}{3}$.

2159. Any iteration which is periodic, but has no fixed points, will provide a counterexample. One such is $x_{n+1} = 1 - \frac{1}{x_n}$, which has period 3.

— Alternative Method —

Let f be a function defined over the domain $\{0, 1\}$, with $\mathrm{f}(0) = 1$, $\mathrm{f}(1) = 0$. Then the iteration $u_{n+1} = \mathrm{f}(u_n)$ has no fixed points. But, running it with $u_n = 0$ gives $0, 1, 0, 1, ...$, which is periodic.

2160. The first derivative is $\frac{dy}{dx} = 2x - x^{-2}$. Substituting this and our expression for y into the LHS, we get

$$\begin{aligned} & 2x - x^{-2} + \frac{x^2 + \frac{1}{x}}{x} \\ \equiv\ & 2x - x^{-2} + x + x^{-2} \\ \equiv\ & 3x. \end{aligned}$$

So, the curve satisfies the DE. Furthermore, since $1^2 + \frac{1}{2} = 2$, the curve passes through $(1, 2)$.

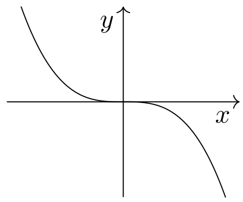
2161. (a) Since the y axis is a line of symmetry, areas either side of the y axis are equal. Hence, the integral has value $k + k = 2k$.

(b) We can split the integral up. The first part has value k. We can then simplify the second part using the fact that, due to the symmetry, $\mathrm{f}(-x) = \mathrm{f}(x)$ for all x:

$$\int_0^1 -\mathrm{f}(-x)\,dx = \int_0^1 -\mathrm{f}(x)\,dx = -k.$$

So, the original integral gives $k - k = 0$.

2162. Any graph with the broad shape of the negative cubic $y = -x^3$ will do the trick. The graph should have no roots or stationary points other than at $x = 0$. The curve $y = -x^3$ is shown:



2163. We know that $\lambda_1 = 1 - \lambda_2$. So, the point P has position vector $\mathbf{p} = (1 - \lambda_2)\mathbf{a} + \lambda_2\mathbf{b}$, which can be rearranged to

$$\begin{aligned} \mathbf{p} &= \mathbf{a} + \lambda_2(\mathbf{b} - \mathbf{a}) \\ &= \overrightarrow{OA} + \lambda_2\overrightarrow{AB}. \end{aligned}$$

So, P must lie on the line AB. And, since both constants are positive, we know that $\lambda_2 \in (0, 1)$, which puts P between A and B. □

2164. (a) False. A counterexample is $x = y = \frac{2}{3}$.

(b) False. A counterexample is $x = \frac{1}{3}$, $y = \frac{2}{3}$.

(c) True.

2165. Assume, for a contradiction, that $a, b, c \in \mathbb{N}$ are a Pythagorean triple, where a, b, c have no common factors and c is even. Since $a^2 + b^2 = c^2$, this means that either a and b are both even, or both odd.

- If a and b are both even, then the triple has a common factor of 2.
- If a and b are both odd, then, for $p, q, r \in \mathbb{N}$,

$$\begin{aligned} &(2p+1)^2 + (2q+1)^2 = (2r)^2 \\ \Longrightarrow\ & 4p^2 + 4p + 1 + 4q^2 + 4q + 1 = 4r^2. \end{aligned}$$

  The LHS is two more than a multiple of 4, while the RHS is a multiple of 4.

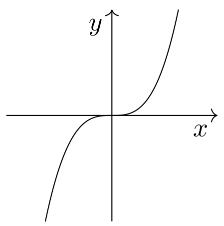
In both cases, we have a contradiction. Hence, in a primitive Pythagorean triple (a, b, c), c cannot be even. □

2166. This is implicit differentiation, which is simply an example of the chain rule. If, to be explicit, we define $z = y^2$, then the chain rule tells us that

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}.$$

This is the required result.

2167. This is true of a cubic with two stationary points (the classic cubic shape), but not of one with fewer. So, $y = x^3$ is a counterexample. It has its centre of rotational symmetry at the origin.



The normal there is the y axis, which does not intersect the curve again.

2168. Using the binomial distribution, the equation is

$$^4\mathrm{C}_1 p^1 (1-p)^3 \approx {}^4\mathrm{C}_2 p^2 (1-p)^2.$$

Now, since $0 < p < 1$, we know that neither p nor $(1-p)$ is equal to zero. Hence, we can divide through by $p(1-p)^2$, giving $4(1-p) \approx 6p$. Solving this gives $p \approx 2/5$.

2169. Differentiating implicitly by the chain rule,

$$\begin{aligned} &\tfrac{d}{dx}(x+y) \times \tfrac{d}{dy}(x-y) \\ \equiv\ &\left(1 + \tfrac{dy}{dx}\right)\left(\tfrac{dx}{dy} - 1\right) \\ \equiv\ &\tfrac{dx}{dy} - \tfrac{dy}{dx} - 1 + \tfrac{dy}{dx} \times \tfrac{dx}{dy} \\ \equiv\ &\tfrac{dx}{dy} - \tfrac{dy}{dx} - 1 + 1 \\ \equiv\ &\tfrac{dx}{dy} - \tfrac{dy}{dx}. \end{aligned}$$

— Nota Bene —

The step from line three to line four relies on the fact that $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are reciprocals.

- At first glance, this is obvious.
- At second glance, it isn't obvious: these are limits, not literally fractions.
- At third glance, it is obvious again: since these limits are the limits of fractions $\frac{\delta y}{\delta x}$, they act as fractions do.

2170. The parabola is stationary on the y axis, meaning that its vertex lies on $x = 0$. Hence, it must have an equation of the form $y = px^2 + q$. So, neither a nor c can be determined, but we do know that $b = 0$.

2171. In general, $\mathrm{f}(|x|) \equiv \mathrm{f}(x)$ is true if the function f is even, i.e. if the graph $y = \mathrm{f}(x)$ has the y axis as a line of symmetry. That way $\mathrm{f}(x) = \mathrm{f}(-x)$, so the mod sign doesn't make any difference. Of the main trig functions, only the cosine function has this symmetry. Reciprocating the outputs doesn't affect input symmetry, so $\sec|x| \equiv \sec x$ is the only valid identity:

(a) False,
(b) True,
(c) False.

2172. A counterexample is $\mathrm{g}(x) = -x^3$, defined over $\mathbb{R}$. This is polynomial, and, for any positive k, we have $-k^3 < 0$. However, the gradient at $x = 0$ is $-3x^2\big|_{x=0} = 0$.

2173. The other factor is linear: $(ax+b)$. The coefficient of x^3 tells us that $a = 5$ and the constant term tells us that $b = -1$. Hence, the expression has roots if either $x^2 + x + 1 = 0$ or $5x - 1 = 0$. But the discriminant of the quadratic is $\Delta = -3 < 0$, so it has no real roots. The only real root, therefore, is $x = 1/5$.

2174. The number of cards in each storey is given by $c_1 = 2, c_2 = 5, c_3 = 8, \ldots$. This is an AP with first term $a = 2$ and common difference $d = 3$. Hence, the sum over n storeys is

$$\begin{aligned} S_n &= \tfrac{1}{2}n\big(2a + (n-1)d\big) \\ &= \tfrac{1}{2}n(4 + 3n - 3) \\ &\equiv \tfrac{1}{2}n(3n+1), \text{ as required.} \end{aligned}$$

2175. The first point gives us $A = 3$. Then the second gives $3e^k = 4$, so $k = \ln 4/3$. The curve which fits the first two points, then, is $y = 3e^{x \ln 4/3}$. At $x = 2$, we get $y = \frac{16}{3} = 5.\dot{3}$. This is nowhere near $y = 10$. So, the model is inappropriate.

—— Alternative Method ——

Such an exponential model would require, since the x values increase in AP, that the y values increase in GP. But $4/3 \neq 10/4$. So, an exponential model is inappropriate.

2176. Integrating by substitution, we let $u = 1 + 4x$. Then $\frac{du}{dx} = 4$, so $dx = \frac{1}{4}du$. Also $60x = 15(u-1)$. We can now enact the substitution:

$$\begin{aligned}&\int_{x=0}^{x=1} 60x\sqrt{1+4x}\,dx\\ =&\int_{u=1}^{u=5} 15(u-1)\sqrt{u}\,\tfrac{1}{4}du.\end{aligned}$$

Multiplying out and simplifying, this is

$$\begin{aligned}&\tfrac{15}{4}\int_1^5 u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du\\ =&\ \tfrac{15}{4}\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right]_1^5\\ =&\ \left(\tfrac{3}{2}\cdot 25\sqrt{5} - \tfrac{5}{2}\cdot 5\sqrt{5}\right) - \left(\tfrac{3}{2} - \tfrac{5}{2}\right)\\ =&\ 1 + 25\sqrt{5}.\end{aligned}$$

—— Alternative Method ——

By parts, let $u = 60x$ and $v' = \sqrt{1+4x}$. Then $u' = 60$ and, by the reverse chain rule,

$$\begin{aligned}v &= \tfrac{2}{3}(1+4x)^{\frac{3}{2}} \times \tfrac{1}{4}\\ &= \tfrac{1}{6}(1+4x)^{\frac{3}{2}}.\end{aligned}$$

The parts formula is

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Substituting in,

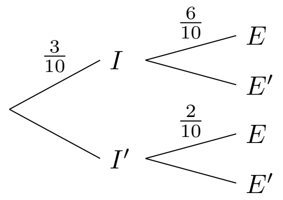
$$\begin{aligned}&\int_0^1 60x\sqrt{1+4x}\,dx\\ =&\left[60x\cdot\tfrac{1}{6}(1+4x)^{\frac{3}{2}}\right]_0^1 - \int_0^1 60\cdot\tfrac{1}{6}(1+4x)^{\frac{3}{2}}\,dx\\ =&\left[10x(1+4x)^{\frac{3}{2}}\right]_0^1 - \int_0^1 10(1+4x)^{\frac{3}{2}}\,dx.\end{aligned}$$

Integrating by the reverse chain rule, this is

$$\begin{aligned}&\left[10x(1+4x)^{\frac{3}{2}} - 10\cdot\tfrac{2}{5}(1+4x)^{\frac{5}{2}}\cdot\tfrac{1}{4}\right]_0^1\\ =&\left[10x(1+4x)^{\frac{3}{2}} - (1+4x)^{\frac{5}{2}}\right]_0^1\\ =&\left(10\times 5\sqrt{5} - 25\sqrt{5}\right) - (-1)\\ =&\ 1 + 25\sqrt{5},\ \text{as required.}\end{aligned}$$

2177. For vertical asymptotes, the denominator must be zero, so we require $x^{2k+1} = b^{2k+1}$. Since $2k+1$ is an odd number, this has exactly one real root at $x = b$. This is the equation of the only vertical asymptote.

2178. (a) The tree diagram is



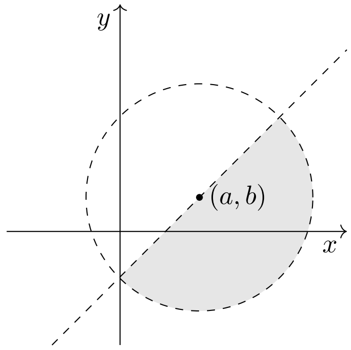
(b) We multiply along the first and third branches, before adding. This gives

$$\mathbb{P}(E) = \tfrac{3}{10}\times\tfrac{6}{10} + \tfrac{7}{10}\times\tfrac{2}{10} = 0.32.$$

(c) We restrict the possibility space to the first and third branches. This gives

$$\mathbb{P}(I \mid E) = \frac{\frac{3}{10}\times\frac{6}{10}}{\frac{3}{10}\times\frac{6}{10}+\frac{7}{10}\times\frac{2}{10}} = \tfrac{9}{16} \approx 56\%.$$

2179. The inequalities are those of $y < x$ and $x^2 + y^2 < 1$, translated by vector $a\mathbf{i} + b\mathbf{j}$. So, the region is



2180. The derivative is

$$\frac{dy}{dx} = 1 - \tfrac{1}{3}x^{-\frac{2}{3}}.$$

Evaluating this at $x = -1$, we get $m_{\text{tangent}} = \frac{2}{3}$. Substituting the point $(-1, 0)$, the equation of the tangent is

$$y = \tfrac{2}{3}x + \tfrac{2}{3}.$$

Solving simultaneously for the re-intersection:

$$\begin{aligned}&x - x^{\frac{1}{3}} = \tfrac{2}{3}x + \tfrac{2}{3}\\ \Longrightarrow\ &x - 3x^{\frac{1}{3}} - 2 = 0.\end{aligned}$$

This is a cubic in $x^{\frac{1}{3}}$. So, let $z = x^{\frac{1}{3}}$, which gives $z^3 - 3z - 2 = 0$. We know this to have a double root (point of tangency) at $x = -1$, which is at $z = -1$. So, $(z+1)^2$ must be a factor. This gives $(z+1)^2(z-2) = 0$. Hence, the re-intersection is at $z = 2$, so $(x, y) = (8, 6)$.

2181. We cannot take the limit as it stands, because the numerator and denominator are both zero at $x = 4$. But we can factorise, giving

$$\lim_{x \to 4} \frac{(x-4)(x+4)}{(x-4)(x+5)}.$$

Since $x \neq 4$, we can divide top and bottom by $x - 4$, at which point it is safe to take the limit:

$$\lim_{x \to 4} \frac{x+4}{x+5} = \tfrac{8}{9}.$$

2182. Using index laws, we can rewrite as

(a) $\sqrt{e^{3x}} \equiv (e^x)^{\frac{3}{2}}$,
(b) $\sqrt{e^{3x+1}} \equiv (e^x)^{\frac{3}{2}} \times e^{\frac{1}{2}}$,
(c) $\sqrt{e^{3x+2}} \equiv (e^x)^{\frac{3}{2}} \times e$.

2183. The possibility space is

A \ B	1	2	3	4	5	6
1					✓	
2				✓		✓
3			✓		✓	
4		✓		✓		
5	✓		✓			
6		✓				

The successful outcomes are denoted with ticks. The fact "A is prime" restricts the possibility space to the shaded region. The relevant calculations are

$$\mathbb{P}(|A - B| = 1) = \tfrac{10}{36} = \tfrac{5}{18},$$
$$\mathbb{P}(|A - B| = 1 \text{ given } A \text{ prime}) = \tfrac{6}{18}.$$

Knowing A to be prime increases $\mathbb{P}(|A - B| = 1)$.

2184. (a) In general, this will not have two real roots. For example, $x^3 + 4x^2 + 4x = 0$ has two real roots, but $2x^3 + 5x^2 + 5x = 0$ has only one.

(b) This must have exactly two real roots. If the original equation had roots $0, \alpha$, then this new one has roots $2x - 3 = 0, \alpha$, i.e. $x = \frac{3}{2}, \frac{\alpha+3}{2}$.

(c) This must have exactly two real roots. All that has happened is that the single root at $x = 0$ (factor of x) has become a double root at $x = 0$ (factor of x^2). The overall number of roots is unchanged.

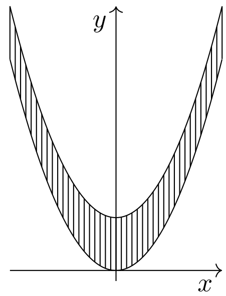
2185. The region in question is sandwiched between the parabolae $y = x^2$ and $y = x^2 + 1$. This is a curved strip of constant y height. Integrating this between $x = -p$ and $x = p$, and letting p become large, the area in question is

$$\lim_{p \to \infty} \int_{-p}^{p} 1 \, dx = \lim_{p \to \infty} 2p.$$

This grows without limit, so the area is infinite.

— Nota Bene —

Visually, the integral is adding up the lengths of the vertical lines in the diagram below.



The length of every vertical line is 1. So, the area is the same as a $1 \times \infty$ rectangle, i.e. infinite.

2186. A number ends 00 iff it has 10^2 as a factor, which means two factors of 2 and two factors of 5. The product of nine consecutive integers, e.g. 9!, can't be guaranteed to contain two factors of 5, but ten consecutive integers can. This will also have two (and more) factors of 2. So, $n = 10$.

2187. (a) At 90°, the child goes straight up, so $d = 6$.

(b) At 0°, the child begins 3 m above the water. Vertically, $s = 3$, $u = 0$ and $a = 10$. Hence, $3 = 5t^2$, which gives $t = \sqrt{3/5}$. Horizontally,

$$d = 3 + \sqrt{3/5} \times \sqrt{60} = 9.$$

(c) At 60°, initial height is $6 - 3\cos 60° = 4.5$, and horizontal displacement is $3 \sin 60° = \sqrt{3}/2$. Initial speeds are

$$u_x = \tfrac{\sqrt{3}}{3} \cos 60° = \tfrac{\sqrt{3}}{6},$$
$$u_y = \tfrac{\sqrt{3}}{3} \sin 60° = \tfrac{1}{2}.$$

Vertically, $s = -4.5$, $u = \frac{1}{2}$ and $a = -10$:

$$-4.5 = \tfrac{1}{2}t - 5t^2$$
$$\Longrightarrow t = -0.9, 1.$$

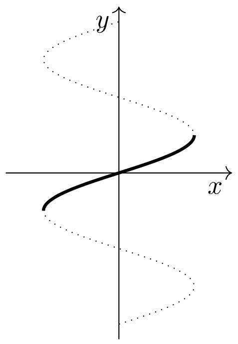
We reject the negative time. Hence, the total horizontal distance travelled is

$$d = 3 + 3\sin 60° + \tfrac{\sqrt{3}}{6} \times 1$$
$$= 3 + \tfrac{5\sqrt{3}}{3}.$$

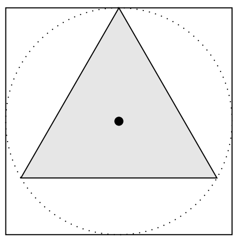
2188. The sample $\{(x_i, y_i)\}$ can be thought of as a set of points in an (x, y) plane. The y coordinates of these points are being mapped by $y_i \mapsto ay_i + b$. This is a stretch, scale factor a, in the y direction, followed by a translation by b units in the positive y direction.

The correlation coefficient r measures closeness to a *linear* relationship. Since neither a stretch nor a translation has any affect on such closeness, the value of r will be unaffected.

2189. Most of this is correct. However, to construct an inverse, one must first restrict the domain and codomain, to produce a one-to-one function. So, while the graph $x = \sin y$, shown dotted below, is a full sine wave, the graph $y = \arcsin x$, shown solid, is only a finite part of it:



2190. We need the centre of the equilateral triangle to lie at the centre of the square, with one of the vertices of the triangle lying at the midpoint of one of the sides of the square, as shown:



Here, we use the standard result that the centroid of a triangle divides the medians in the ratio $1 : 2$. This puts the perpendicular height of the triangle (vertical in the diagram) at $\frac{3}{4}$.

— Nota Bene —

The *centroid* of a triangle is the point where the medians meet. A *median* is a line joining a vertex to the midpoint of the opposite side.

2191. Call the function f. Multiplying out,

$$\begin{aligned} \mathrm{f}(x) &= e^x - e^{2x} \\ \Longrightarrow \mathrm{f}'(x) &= e^x - 2e^{2x} \\ \Longrightarrow \mathrm{f}''(x) &= e^x - 4e^{2x}. \end{aligned}$$

At points of inflection, $\mathrm{f}''(x) = 0$ is zero:

$$\begin{aligned} e^x - 4e^{2x} &= 0 \\ \Longrightarrow e^x(1 - 4e^x) &= 0. \end{aligned}$$

The exponential e^x is always positive, so $x = \ln \frac{1}{4}$. Substituting this back in,

$$\begin{aligned} \mathrm{f}\big(\ln \tfrac{1}{4}\big) &= e^{\ln \frac{1}{4}} - e^{2 \ln \frac{1}{4}} \\ &= e^{\ln \frac{1}{4}} - e^{\ln \frac{1}{16}} \\ &= \tfrac{1}{4} - \tfrac{1}{16} \\ &= \tfrac{3}{16}, \text{ as required.} \end{aligned}$$

2192. (a) Differentiating, $\dfrac{du}{dx} = 2x$.

(b) The chain rule, with x as the value of the inside function, gives

$$\frac{d}{du}\big(x^3\big) \equiv 3x^2 \times \frac{dx}{du} \equiv 3x^2 \div \frac{du}{dx}.$$

(c) Combining the results from (a) and (b),

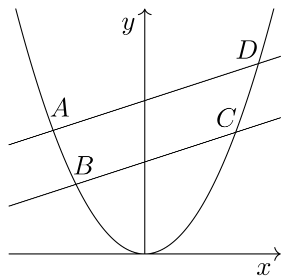
$$\begin{aligned} \frac{d}{d\big(x^2\big)}\big(x^3\big) &\equiv \frac{d}{du}\big(x^3\big) \\ &\equiv 3x^2 \div \frac{du}{dx} \\ &\equiv 3x^2 \div (2x) \\ &\equiv \tfrac{3}{2}x. \end{aligned}$$

2193. (a) i. The first card can be anything, leaving 12 of the same suit. So, $\mathbb{P}(\text{same suit}) = \frac{12}{51}$.

ii. There is only one card of a given suit and number, so the fact that the cards are the same suit precludes them being the same number. Hence, the probability is zero.

(b) Since the answers in i. and ii. are not equal, the events "same suit" and "same number" are not independent.

2194. Assume, for a contradiction, that distinct points A, B, C, D on $y = x^2$ form a rectangle. Let the x coordinates of these points be $a < b < c < d$. Consider parallel sides AD and BC, intersecting with the curve as shown below:



We know that $d - a > c - b$. Hence, since the sides AD and BC are parallel, $|AD| > |BC|$. But, since the shape is a rectangle, $|AD| = |BC|$. This is a contradiction. Hence, no set of four distinct points on $y = x^2$ form a rectangle. □

2195. The exponential and natural logarithm functions are inverses. And both are one-to-one by original definition. So, $y = e^x$ and $x = \ln y$ are the same statement: $y = e^x \iff x = \ln y$.

2196. Factorising, we get $y = x^3(x + 1)^3$, which should have triple roots at $x = 0$ and $x = -1$. But the graph shown has triple roots at $x = \pm 1$. So, the graph could be $y = (x - 1)^3(x + 1)^3$, but not the equation given.

2197. We use the formula $\cos 2\theta \equiv 1 - 2\sin^2\theta$, with $2x$ playing the role of θ:

$$\begin{aligned} & \sin^2 2x + \cos 4x = 0 \\ \Longrightarrow\ & \sin^2 2x + 1 - 2\sin^2 2x = 0 \\ \Longrightarrow\ & \sin^2 2x = 1 \\ \Longrightarrow\ & \sin 2x = \pm 1. \end{aligned}$$

Writing the solution longhand to begin with,

$$\begin{aligned} & 2x = ..., \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{5\pi}{2}, \tfrac{7\pi}{2} ... \\ \Longrightarrow\ & x = ..., \tfrac{\pi}{4}, \tfrac{3\pi}{4}, \tfrac{5\pi}{4}, \tfrac{7\pi}{4} ... \\ \Longrightarrow\ & x = \left(\tfrac{1}{2}n - \tfrac{1}{4}\right)\pi, \text{ for } n \in \mathbb{Z}. \end{aligned}$$

2198. (a) From a calculator, the binomial probability is 0.39174.... The normal approximation, then, is $Y \sim N(np, npq)$, i.e. $Y \sim N(2, 1.8)$. Using a continuity correction, the normal value for probability is $\mathbb{P}(-0.5 \le Y \le 1.5) = 0.32349...$ The percentage error is

$$\frac{0.32349 - 0.39174}{0.39174} \approx -17\%.$$

(b) This large percentage error stems from the fact that the normal approximation doesn't hold when either np or npq is small (say less than 10). In this case $np = 2$.

— Nota Bene —

The need for a continuity correction stems from the fact that binomials are discrete distributions, while normals are continuous. Hence, the original inequality $0 \le X \le 1$ is $X \in \{0, 1\}$. The normal inequality becomes $-0.5 \le Y \le 1.5$, including all values of the approximating Y distribution which round to 0 or 1.

2199. Translating back into regular algebra:

$$\begin{aligned} & \text{Int}(x^2 - k) = \text{Int}(x^4) \\ \Longrightarrow\ & \int_0^1 x^2 - k\,dx = \int_0^1 x^4\,dx \\ \Longrightarrow\ & \left[\tfrac{1}{3}x^3 - kx\right]_0^1 = \left[\tfrac{1}{5}x^5\right]_0^1 \\ \Longrightarrow\ & \tfrac{1}{3} - k = \tfrac{1}{5} \\ \Longrightarrow\ & k = \tfrac{2}{15}. \end{aligned}$$

2200. We can divide top and bottom by x, at which point the small fractions over x tend to zero and it is safe to take the limit:

$$\begin{aligned} & \lim_{x\to\infty} \frac{2x+3}{3x+2} \\ = & \lim_{x\to\infty} \frac{2 + \frac{3}{x}}{3 + \frac{2}{x}} \\ = & \tfrac{2}{3}, \text{ as required.} \end{aligned}$$

— End of 22nd Hundred —